On the Forms of Sextic Scrolls having a Rectilinear Directrix.

By Virgil Snyder.

1. In Volume XXV of the Journal I have three papers on sextic scrolls, pp. 59–84, 85–96, 261–268, wherein 118 types of the surface are discussed and enumerated. Besides the references there given, three papers were in existence which treat exclusively or partly of sextic scrolls. The first of these is the doctor dissertation of Dr. Karl Fink, "Ueber windschiefe Flächen im allgemeinen und insbesondere über solche des sechsten Grades," Tübingen, 1886. The first few pages are devoted to a general discussion of the correspondence between the points of any two plane sections of scrolls, including the formulas which I have given on p. 75 and on p. 268, but various false conclusions are drawn from the results. The principles thus established are applied to the S_6, but so carelessly that over half the types mentioned are impossible, and a large number of others exist of which no mention is made. The paper is practically worthless.

The second paper is the doctor dissertation of Dr. Jakob Bergstedt, "Om regelytor af sjette graden, I, unikursale ytor,'' Lund., 1886. The abstract of this memoir in the Fortschritte der Mathematik is rather misleading. By no means all of the forms of unicursal sextics are derived, yet the number is over 60, the equations of many being derived. The paper has just about the same degree of completeness as my first paper, and nearly the same methods are employed, except that Bergstedt more systematically analyzes the configurations of the multiple points on the nodal curve and uses correspondence but a few times. Several false conclusions are arrived at. These two papers would not materially affect the truth of the statement that no (complete) systematic dis-

cussion on S_6 exists. The third paper, however, the doctor dissertation of Professor Anders Wiman, "Klassifikation of regelytorna af sjette graden," Lund., 1892, is of very different nature. The memoir is not even mentioned in the Fortschritte, yet it contains 111 pages and enumerates 118 types of the surface. The method employed by Wiman is radically different from any of the preceding. He establishes a (1, 1) correspondence between the points of space and the lines of a complex, by means of which the surface becomes a twisted curve. But few equations are given, the existence of the various types being established by geometric methods. In some cases the equations can be easily obtained, while in others the difficulty is shifted to the twisted curve, satisfying prescribed conditions.

In the present paper I wish to complete the enumeration, including the derivation of the equation, of those scrolls which have a rectilinear directrix, and to correct the errors in my previous results. The same methods will be employed as were used in the preceding papers. The purpose is to complete Wiman's results by deriving the equations, and to add those which he overlooked.

A.—*Unicursal Scrolls.*

§1. *Simple directrix line.*

2. As was shown, p. 76 (of my own paper), type XXX, a S_6 with a simple directrix line c_1^1, and c_{10}^2 with a multiple point, through which pass 4 generators, exists. This point is a sixfold point on c_{10}, which, consequently, lies on a cone of order 4, k_4. B., p. 19, proves that k_4 is of genus $p=3$. Suppose, now, c_1 pierces the plane of c_5 in a point on a double tangent. The points of intersection of the tangent and c_5 are consecutive, hence the plane contains two pairs of torsal generators and a simple generator. The former intersect in a double point on c_{10}^2, the tangents at which are the torsal generators. Similarly, c_1^1 may lie on two or three double tangents, giving rise to two or three further double points on c_{10}, the latter being unicursal.

Finally, if the point of intersection be on four double tangents, the nodal curve will be composite. Let c_5 be defined by

$$(x^4+y^4)(y+ax)-(y^2+yz+bz^2)(y+dz)=0, \quad w=0,$$

wherein

$$a - d - 2 = 0. \quad 2b + 2d + 1 = 0, \quad 2b + d + 2bd = 0,$$

and let c_1 be $x = 0$, $y = 0$. The twisted quintics will each have a triple point at (1, 0, 0, 0) and will intersect in four other points. Each curve will lie on a quadric cone. When four generators are concurrent, the only other case is that of a fivefold line, which was exhaustively discussed in types I to XII of my first paper.

3. Now establish a similar (1, 1) correspondence between c_1 and c_5, the latter being unicursal and having a triple point. The resulting c_{10}^2 on S_6 has four triple points and is of genus 3 (B., p. 21). Here again 1, 2, 3 or 4 double points may appear, the last case necessitating that c_{10} break up into two c_5, each having two double points through which the other curve passes singly. Let A be one of these points. The cones, having A for vertices, are of order 3 for c_5 and of order 4 for c_5'. The former has one double generator which is a single generator of the latter, and the latter has three double generators, two of which are single on the former. The cones still have six common generators. Consider the plane passed through c_1 and A. It will contain five generators to S_6, three of which pass through A. Each generator must contain two points on each c_5. From a figure, it is easily seen that two of these generators of S_6 are common generators to the two cones, while the third is a double generator to k_4. This accounts for two further intersections of the cones, hence c_5 and c_5' have four actual or apparent points of intersection. It was seen above that they have four actual intersections, at all of which the tangents to each intersect c_1. A simple directrix may also be a simple or a multiple generator, but as the order of the residual nodal curve is correspondingly lower, such cases will be considered in connection with multiple directrices. Wiman does not distinguish between them.

§2.—*Double directrix line.* (a). *No double generators.*

4. Besides XIX, in which c_9 is of genus 1, another form exists by letting c_1 lie on a double tangent, as another double point appears. If c_1 lies on two double tangents, c_9 breaks up into c_6 having a fourfold point and a plane c_3 having its double point at the same point. The two curves have two points of intersection. If c_1 lies on three double tangents, c_6 breaks up into two plane

cubics having a common point for node. Each pair intersect in one further point.

5. By establishing a (1, 2) correspondence between a c_5 with a triple point and a c_1 which passes through a simple point as self-corresponding point, the S_6 has a c_9 with four triple points and of genus 1. The double point may appear as above. If two double points appear, c_9 breaks up into a c_6 with four double points and a twisted c_3 passing through each double point and two simple points on c_6. c_1 cuts c_3 once and c_6 twice. When c_1 lies on three double tangents, c_6 breaks up into two c_3 which pass through the four nodes, intersect each other in one point and each intersects the first c_3. Each c_3 cuts c_1 once. If, instead of a (1, 2) correspondence between c_1 and c_5 with a self-corresponding point at their intersection, we establish a general (1, 1) correspondence, these eight forms appear again, except that c_1 is now simple directrix and simple generator. Four of these were given in forms XXXI–XXXIV.

(b). *One Double Generator.*

6. If, in the (1, 2) correspondence between c_5 and c_1, the two values of the parameter on a node should correspond to the point on c_1, the line joining them is a double generator g^2. The six types analogous to (a) are found, but the symbol $c_1^2 + c_6^2 + c_3^2$ can break up into $c_1^2 + c_5^2 + c_3^2 + g^2$ or $c_1^2 + c_6^2 + c_2^2 + g^2$ in both systems, those arising from a plane quintic directrix having a fourfold point and those arising from a plane unicursal quintic directrix having a triple point. By accounting for a sixfold point and the required number of nodes in the first case and four triple points and appropriate nodes in the second, the lines have the following configurations: $c_1^2 + c_8^2 + g^2$. The c_8 has a fivefold point. The g^2 cuts c_8 in the fivefold point and one simple point. c_1 cuts c_8 in two points $c_1^2 + c_6^2 + c_2^2 + g^2$.

c_6 has a fourfold point, passing through g^2. c_2^2 passes through the fourfold point and cuts g^2 in one other point and c_6 in two simple points, c_1 cuts c_6 twice. This is derived from type LXIII, eq., p. 90, by making g^2 in plane of c_2. $c_1^2 + c_5^2 + [c_3^2] + g^2$. c_5 has a triple point at node of c_3, the two curves having two points of simple intersection. g^2 passes through the multiple point and one

other point of c_5. c_1 cuts each curve once. Let

$$\begin{array}{llll} x = 1 - \lambda^2, & y = \lambda(1 - \lambda^2), & z = 0, & w = 1 \text{ be } c_3, \\ x = 1, & y = 0, & z = 0, & w = 1 \text{ be } c_1, \end{array}$$

and let λ, μ be connected by a correspondence of the form

$$\lambda^2 \left(4a(a + c)\mu^2 + 4a'(a + c)\mu + a'^2\right) + 4c(a + c)\mu^2 = 0.$$

The resulting S_6 is of the form required.

For the case $c_1^2 + c_2^2 + 2[c_3^2] + g^2$ there is no ambiguity, and the equation is derived directly from the general case.

7. In the second case, the c_8 has two triple points and two double points, g^2 passing through both of the latter and one simple point. c_8 cuts c_1 in two points, and is of genus one. One double point may appear when c_1 lies on a double tangent. For the case $c_1^2 + c_6^2 + c_2^2 + g^2$, c_6 has two double points through which g^2 passes, and two others, through which passes c_2. The latter cuts c_6 in two simple points and g^2 once. c_1 cuts c_2 twice. The two cases of the same symbol may be easily distinguished by the position of g^2. In the [2, 4] case, it lies in the plane of c_2, while in the [3, 3] case it does not. The symbol $c_1^2 + c_5^2 + c_3^2 + g^2$ differs from the preceding by having a twisted c_3. The c_5 has three simple points on g^2, it also has two double points, through each of which c_3 passes. c_5 and c_3 each cut c_1 in one point $c_1^2 + 2c_3^2 + c_2^2 + g^2$; both of the c_3 are twisted. They intersect in four points, through two of which passes c_2 and through the other two g^2. The two c_3 intersect in one other point; c_2 intersects each c_3 and g^2 in one point besides the common intersection. c_1 cuts each c_3 once.

(c). *Two Distinct Double Generators.*

8. In the same manner, as in (b), introduce two double generators, intersecting in the fourfold point of c_5. The residual is a c_7^2 having a fourfold point at the intersection of the double generators, and one point on each g^2 as well as on c_1, $p = 1$, and can be made zero in the usual way.

If the two nodes on c_5 are consecutive, the $2g^2$ can become tacnodal or consecutive. This scroll can also be generated as follows: Given a tacnodal c_4 with

one other double point. Take any line cutting the tacnodal tangent as directrix in a (1, 2) correspondence with c_4, the point of intersection being a pinch-point. The tacnodal tangent will become a tacnodal generator. The equations of c_4 may be

$$w = 0, \quad (yz - mx^2)(yz - m'x^2) = fx^3y + gx^2y^2.$$

The point (0, 0, 1, 0) is the tacnode, with $y = 0$, $w = 0$, for tangent (0, 1, 0, 0) is a crunode. Let c_1 be $y = 0$, $z = 0$, $w = k$, $x = 1$. It is projective with the pencil $x = kz$, $w = 0$. Connect the two points in which a line of this pencil cuts c_4 with the corresponding point on c_1. The S_6 is the k eliminant of $k^2z = kx - w$ and

$$[y - m(kx - w)][y - m'(kx - w)] = fy^2k^3 + gk^2y^2.$$

The plane $y = 0$ contains the double directrix and the generator $w = 0$ as a four-fold line. $p = 1$ or 0.

9. Similarly for the [3, 3] case. c_7 now has four double points, two of which lie on each g^2. Each g^2 cuts c_7 in one simple point. $p = 1$ or 0. In case of the tacnodal generators, c_7 has two triple points. It intersects $\overline{2g^2}$ in four points, two of them having their tangents in the plane of the singular g^2. If, in the [2, 4] case, c_1 lies on two double tangents, the symbol becomes $c_1^2 + c_4^2 + [c_3^2] + 2g^2$, the double generators being distinct. c_4 has a node in common with $[c_3]$ and two simple points in common with it. c_3 cuts c_1 once. No further forms can exist when the double generators are distinct. When they are consecutive, the first new type becomes $c_1^2 + c_5^2 + c_2^2 + \overline{2g^2}$. The S_6 may be generated as follows:

Let c_2 be $\quad w = \lambda, \quad x = \lambda^2, \quad y = 1, \quad z = 0,$

and c_1 be $\quad x = 0, \quad y = 0, \quad z = \mu, \quad w = 1.$

If $\lambda = 0$, $\mu = 0$ be a double element in the (2, 2) correspondence between λ, μ, the line $x = 0$, $z = 0$ will be a tacnodal generator. c_5 has a triple point, c_2 touching one branch, with g^2 as tangent. c_5 and c_2 have one other point in common. Finally, if c_1 lies on three double tangents, the symbol becomes $c_1^2 + [c_3^2] + 2c_2^2 + \overline{2g^2}$. Consider the conics:

$$c_2, \quad x = 1, \quad y = \mu, \quad z = 0, \quad w = \mu^2,$$
$$c_2', \quad x = 0, \quad y = \lambda, \quad z = 1, \quad w = \lambda^2,$$

which touch each other at (0, 0, 0, 1) with $x=0$, $z=0$ for common tangent. The equations of the line joining λ to μ are

$$\mu x+\lambda z-y=0, \quad \mu^2 x+\lambda^2 z-w=0.$$

Any line cutting the common tangent is of the form

$$bx-z=0, \quad cx+y+aw=0.$$

A generator will cut this line when

$$ab\lambda^2+a\mu^2+b\lambda+\mu+c=0.$$

The equation of S_6 becomes

$$\begin{vmatrix} a(z-bx) & z-bx & abw+by+cz & 0 \\ 0 & a(z-bx) & z-bx & abw+by+cz \\ x(x+z) & -2xy & y^2-zw & 0 \\ 0 & x(x+z) & -2xy & y^2-zw \end{vmatrix} = 0.$$

The section made by the plane $bx-z=0$ is $x^4(aw+by+cx)^2$. The residual nodal curve is of order 3 and cuts every generator once. It cuts c_1 once. The plane $bx-z=0$ cuts c_1 and $x=0$, $z=0$, hence the latter cuts c_3 twice. These points must coincide at (0, 0, 0, 1), since $x=0$, $y^2-zw=0$ define the complete intersection with $z=0$. Hence c_3 has a node at (0, 0, 0, 1). Any plane section through the singular generator will contain a quartic having one double point, and a tacnode at (0, 0, 0, 1).

If the (2, 2) correspondence of the preceding case has a cusp at (0, 0), the result will have this symbol, except that c_3^2 has a cusp at the point of contact. An illustration is afforded by the surface

$$[x(z^2+w^2+xy)+y(z^2-2xz)]^2-4xyw^2(x-z)^2=0,$$

which has a nodal cubic in the plane $w=0$. The c_1^2 is $x=0$, $y=0$, $\overline{2g^2}$ is $x=0$, $z=0$. One double conic is $xy-w^2=0$, $z=0$.

10. If, in the [3, 3] case, c_1 lies on two double tangents, the symbol may be either $c_1^2+c_4^2+c_3^2+2g^2$ or $c_1^2+c_5^2+c_2^2+2g^2$, according as the second g^2 factors off the c_5 or the c_3.

If, between c_3, c_1 as defined in §6, the correspondence be of the form

$$(a + b)\lambda^2\mu^2 = a\lambda^2 + b\mu^2,$$

the residual is a c_4 having three points on each g^2. c_4 cuts c_3 twice. The difference between these forms is, here c_3 is twisted. If the two points on c_2 correspond to the point of intersection of c_1, with its plane in a (2, 2) correspondence, and a double element exists, as

$$(a\lambda^2 + b)\mu^2 + (\lambda^2 + 4bc)(\mu + c) = 0,$$

the two generators will be distinct. The double element is $\lambda = 0$, $\mu = -2c$, while the branch-point is $\mu = 0$, $\lambda = \pm\sqrt{-4bc}$.

The residual curve is c_5 having two double points on one g^2 and three simple points on the other. It cuts c_2 four times, c_2 passing through two intersections of c_5 and g^2.

In case of three double tangents, the symbol becomes $c_1^2 + c_3^2 + 2c_2^2 + 2g^2$. c_3 has two points on each (distinct) g^2 and one on c_1. Each c_2 cuts c_3 in two points on one g^2 and cuts the other g^2 once. The two c_2 have one point in common.

Only two specializations of the tacnodal g^2 can occur; $c_1^2 + c_5^2 + c_2^2 + \overline{2g}^2$. Given $x = \lambda^2$, $y = 1$, $w = \lambda$, $z = 0$; $x = 0$, $y = 0$, $z = \mu$, $w = 1$. The tangent to c_2 at $\lambda = 0$ cuts c_1. The plane containing both is $y = 0$. If the (2, 2) correspondence between λ, μ have $\lambda = 0$, $\mu = \mu_1 \neq 0$ for double element, the line joining 0 to μ is a tacnodal generator not in the plane of the c_2. The c_5 has three points on $\overline{2g}^2$, at two of which it touches the singular plane $y = 0$. Finally, for three double tangents, let $\mu x + \lambda z - \lambda\mu w = 0$, $\lambda x - y + \mu z = 0$ be connected by $\lambda(\mu - 1)^2 + \alpha\mu(\lambda - 1)^2 + \beta(\lambda^2 - \mu^2) = 0$. The plane of the tacnodal g^2 is $2(x + z) - w - y = 0$. This procedure proves that the types LV, LIX and LXI of my enumeration, p. 84, are impossible. In the first case, a surface of type LV must be elliptic and c_7^2 should be replaced by c_5^2. Similarly for LXI, but LIX is interesting from the fact that a S_6 having $2c_2^2$ and c_1^2 must have two double generators to be unicursal. The same result can also be reached by Cayley's method. The method of correspondence cannot be relied upon as a sufficient ground of classification, without interpreting each step geometrically.

Thus, types LV and LXI can be proved impossible by the paper in which they are enumerated, but LIX cannot be thus (directly) explained.

(d). *Three Double Generators.*

11. Three pairs of nodes can only correspond to double generators when one of them is tacnodal. In the case of the plane c_5 has a fourfold point; this requires that all three double generators will be consecutive. The residual curve is a unicursal c_6^2 having a triple point on the singular generator. Any plane section will have an oscnode on the singular generator.

The scroll may be generated as follows:

Given the oscnodal quartic

$$(yw - x^2)^2 = y^3(w - y), \quad z = 0.$$

The line

$$y = kx$$

cuts the quartic in two points distinct from the node. Let this pencil be projective with the range

$$x = 0, \quad y = y, \quad z = k, \quad w = 0.$$

Connect the points of the range with the points in which the corresponding line cuts the quartic. Let the points be denoted by $(x_1, y_1, z_1, 0)$.

$$\begin{gathered}(y_1w_1 - x_1^2)^2 = y_1^3(w_1 - y_1),\\ y_1 = kx_1,\\ xw_1 = x_1w, \quad y_1xk = x_1(ky - z),\end{gathered}$$

hence,

$$\begin{aligned}&k^4x^2 - k^3xw + k^2w^2 - 2kwx + x^2 = 0,\\ &k^2x - ky + z = 0.\end{aligned}$$

The resultant is of the form

$$\begin{vmatrix} 1 & -w & w^2 & -2wx & x^2 & 0 \\ 0 & x & -xw & w^2 & -2wx & x^2 \\ 1 & -y & xz & 0 & 0 & 0 \\ 0 & +1 & -y & z & 0 & 0 \\ 0 & 0 & x & -y & z & 0 \\ 0 & 0 & 0 & x & -y & z \end{vmatrix} = 0.$$

The surface is not contained in a linear congruence. In the [3, 3] case, the singular line is distinct from the ordinary double generator. The residual curve is a c_6^2 having two double points and one simple point on g^2, and four simple points on the singular $\overline{2g^2}$, being touched by the singular torsal plane in two of them. As any plane through c_1^2 contains four generators, c_6^2 does not intersect c_1^2.

If, in the correspondence of §9, the two points in which $z=0$, $y=mx$ cuts the conic, both correspond to $\mu=0$, while each point has $\mu=0$ for pinch-point, the joining line is an ordinary double generator. The residual curve is a rational c_4 having three points on the g^2. To make three consecutive generators, the c_5 should have three coincident cusps. Let

$$x^5 - yw^4 = 0, \quad z = 0$$

be the equations of c_5, and let $x=0$, $y=0$ be the line c_1. In parameters

$$c_5: x = \frac{1}{\lambda}, \quad y = \frac{1}{\lambda^5}, \quad z = 0, \quad w = 1,$$

$$c_1: x = 0, \quad y = 0, \quad z = \mu, \quad w = 1.$$

The point of intersection of c_1, c_5 must be a simple self-corresponding point ($\mu=0$, $\lambda=\infty$). The cusp, $\lambda=0$, must be a double root ($\mu=\infty$, $(\lambda=0)^2$), hence

$$\lambda^2\mu = \lambda - 1.$$

The equation of S_6 is

$$(x+z)^4x^2w^4 - xy(x+w)^4 + 4xyw(x+z)(x+w)^2 - 2xyw^2(x+z)^2 = 0.$$

$x=0$, $y=0$ is c_1^2. $x=0$, $w=0$ is three consecutive double generators, as any plane section will cut this line in an oscnode. In the plane $x+w=0$ lies the double conic

$$(x+z)^2 + xy = 0,$$

which touches the singular generator.

The residual nodal curve is a twisted nodal quartic having at (0, 1, 0, 0) a node. The symbol is $c_1^2 + c_2^2 + c_4^2 + \overline{3g^2}$. Since c_1 intersects c_5 on inflexional tangent, no further forms can appear. Finally, when c_1 lies on two (consecutive) double tangents, the [3, 3] form becomes $c_1^2 + 3c_2^2 + 3g^2$, two g^2 being consecutive. The surface may also be generated as follows: Two conics cut each other

in two points P, Q, and the line joining P, Q is the ordinary g^2. The correspondence between the two conics is (2, 2), having P, Q as simple self-corresponding elements, with the second element at each corresponding to the other. The correspondence is determined by imposing the condition that the line joining corresponding points shall cut a fixed director c_1. Finally, the tacnodal generator must lie in a common tangent plane to the conics through c_1. The correspondence reduces to the form

$$\lambda\mu(\lambda+\mu)+4\lambda\mu+\lambda+\mu=0.$$

The c_1^2 has the equations

$$4z+y+w=0, \quad x=z$$

and the tacnodal generator is

$$x+z+w=0, \quad x+y+z=0.$$

The singular torsal plane is $2z+2x+y+w=0$, which also contains the third double c_2. The equation of S_6 becomes

$$\begin{vmatrix} z-x & -(y+4x) & 1 & x & 0 \\ 0 & z-x & 0 & y+w+4x & x-z \\ wx & x^2-z^2 & y & 0 & 0 \\ 0 & w & 1 & -z & 0 \\ 0 & 0 & z & x^2-wy & zy \end{vmatrix}=0.$$

See Volume XXV, p. 80 ff. of the Journal. Another form of correspondence is $\lambda(\mu-1)^2+\alpha\mu(\lambda-1)^2=0$.

12. The quartic curve

$$(yz-x^2)^2=y^3(z-y), \quad w=0,$$

has an oscnode at (0, 0, 1, 0), $y=0$ being the tangent. Let $y=mx$ cut the curve in two points x_1, z_1, . such that

$$(mz_1-x_1)^2=m^3x_1(z_1=mx_1).$$

Make the lines of the pencil $y = mx$ projective with the range $y = 0$, $z = 0$, $w = mx$. Lines joining points on the range to $x_1 z_1$ will be of the form

$$yz_1 = mx_1 z, \quad y = (mx - w).$$

The equation of the surface becomes

$$(y + w)^4(z^2 - yz + y^2) - 2zx^2 y (y + w)^2 + x^4 y^2 = 0.$$

The line $y = 0, z = 0$ is a double directrix, $y + w = 0$, $x = 0$ is a fourfold directrix. The line $y = 0$, $w = 0$ is an oscnodal or three consecutive generators. The surface has no other nodal lines.

The same configuration will appear whenever the tangent to the oscnode cuts the directrix. The tangent and point of correspondence do not need to be in united position.

Double Contact Directrix.

13. Given a c_5 such that the values of m corresponding to

$$y - \beta = m(x - \alpha),$$

α, β on c_5 form a quartic involution on x. Further, let the four values of x be arranged in two pairs, to the first of which corresponds μ_1, and to the second μ_2. Let m, μ be in (1, 2) correspondence and μ define a point on c_1, $x = \alpha$, $y = \beta$. Finally, if the tangent at α, β correspond to $\mu = 0$ doubly, lines joining corresponding points will generate a S_6. The line c_1 will be double, but any plane section of S_6 not containing c_1 will have a tacnode at the point in which the latter pierces the plane. Any plane through c_1 will contain four generators, two of which pass through μ_1 and two through μ_2. Let α, β be 0, 0. c_5 may be defined by

$$x\,[\phi_2(x, w)]^2 + y\,[\psi_2(x, w)]^2 = 0;$$
$$\phi_2(x, w) = x(ax + a'w); \quad \psi_2(x, w) = bx^2 + b'xw + b''w^2.$$

It has a fourfold point at (0, 1, 0, 0). If

$$\mu^2 = m, \quad y = mx,$$

then

$$[\phi_2(x_1, w_1) + \sqrt{m}\,\psi_2(x_1, w_1)][\phi_2(x_2, w_2) - \sqrt{m}\,\psi_2(x_2, w_2)] = 0.$$

The form of the equation becomes

$$xy\,[ax^2 + a'xy - b'xz - 2b''zw]^2 = y^2\,[\psi_2(x, w)]^2 + 2(b'' - a')\,xyz^2\,\psi_2(x, w) + b''^2x^2z^4.$$

The two values of μ corresponding to it coincide, hence the line $x = 0$, $w = 0$ counts for three consecutive generators. When the two values of μ corresponding to $m = \infty$ do not coincide, there are two distinct double generators in the plane $x = 0$.

In general, corresponding to four double points on c_5 will be four intersections of a μ_1 generator with a μ_2 generator. If c_5 be unicursal, S_6 must, therefore, have $2g^2$. These may be in different planes or the same plane, and in either case, if the tangent cuts c_1, the generator will be singular, counting for two. When the generators are distinct, the residual is a c_6^2 which cuts c_1 twice. In the other case, it becomes a c_5 cutting c_1 once. There are, therefore, four types of unicursal forms in which the residual nodal curve does not degrade. When c_5 is of genus 1, the double generator may be ordinary or singular. All the forms in which the residual nodal curve is composite will be considered separately.

A particular form of this surface can be generated by the forms

$$\lambda^4x + y = 0, \quad \lambda^2w + \lambda y + z = 0.$$

When the double generators do not intersect, the surface can be generated from a c_5 having a triple point. The equations can most easily be obtained from the dual of a later form. Three types exist; two distinct double generators and c_6^2 having two points on the double directrix and two double points on each g^2; an ordinary and a tacnodal generator, the residual being a c_5^2 having two double points on g^2, one point on $\overline{2c_1^2}$ and a double point on $2g^2$ with one branch touching it; finally, two $\overline{2g^2}$ and a c_4^2, which has a singular osculating plane through each singular generator and the directrix. It may be generated by the two cones

$$\lambda^3x - \lambda^2w - z = 0, \quad \lambda^3w + \lambda z + y = 0.$$

The tacnodal directrix is $x = 0$, $y = 0$, and the singular double generators are $x = 0$, $w = 0$; $y = 0$, $z = 0$. Any plane will cut each of these generators in a tacnode. The equation of the scroll may be written

$$(xz^2 + yw^2)^2 + x^2y^2(4zw + xy) = 0.$$

Finally, c_4^2 may break up into $2c_2^2$. Given c_1: $x=0$, $w=0$, $z=\lambda y$, and c_2: $xw-y^2=0$, $z=0$. Pass a plane $x=mw$ through c_1, and let $m(\lambda+1)^2=(\lambda-1)^2$. The plane will cut c_2 in two points $(x_1, y_1, 0, w_1)$. From the equations of a line joining $(0, 1, \lambda, 0)$ to $(x_1, y_1, 0, w_1)$ we obtain

$$\lambda^2(y^2-xw)-2yz\lambda+z^2=0, \quad \lambda^2(x-w)+2\lambda(x+w)+x-w=0,$$

the λ eliminant of which is a S_6 having $2c_1^2$, $x=0$, $y-z=0$ and $w=0$, $y+z=0$ for tacnodal generators, and two double conics.
These forms are not mentioned by Wiman, but he notices the omission, p. 95.

14. Given the conic c_2, $xy-w^2=0$, $z=0$, and the straight line $c_1 x=0$, $y=0$. Establish a (1, 2) correspondence between the planes π through c_1 and the points P_1, P_2 on c_1; let π cut c_2 in A and B. Connect A with P_1 and P_2, and B with P_1 and P_2. c_1 will be a double line and c_2 a double conic on the resulting S_6. The four lines lying in any plane π through c_1 have the peculiar property of meeting c_1 in pairs, hence the last statement made on p. 61, third paragraph, is incorrect.

Let the point $(0, 0, \mu, 1)$ on c_1 be joined to $(x_1, y_1, 0, w_1)$ on c_2.

$$\frac{x}{x_1}=\frac{y}{y_1}=\frac{w\mu-z}{\mu w_1}.$$

Let the plane π contain $(x_1, y_1, 0)$, so that

$$x_1=my_1.$$

Finally, let $m=\dfrac{a'\mu^2+b'\mu+c'}{a\mu^2+b\mu+c}$,

$$m=\frac{x}{y}, \qquad \mu\sqrt{my}=\mu w-z, \qquad \mu^2xy=\mu^2w^2-2\mu wz+z^2,$$

$$\begin{aligned}(ax-a'y)\mu^2+(bx-b'y)\mu+(cx-c'y)&=0,\\ (w^2-xy)\mu^2-2wz\mu+z^2&=0.\end{aligned}$$

$$\begin{aligned}[(w^2-xy)(bx-by)+2wz(ax-a'y)][z^2(bx-b'y)+2wz(cx-c'y)]&\\ +[(cx-c'y)(w^2-xy)-z^2(ax-a'y)]^2&=0.\end{aligned}$$

The line $z=0$, $cx-c'y=0$ is a double generator, and c_1 is a tacnodal directrix. Since the (2, 2) correspondence between c_1 and c_2 has no double element, the S_6 is of genus 1. The residual nodal curve is a c_4 of the first kind cutting c_2 in the two points of intersection with g^2, and in the points (0, 1, 0, 1), (1, 0, 0, 1). The symbol is $\overline{2c_1^2}+c_2^2+c_4^2+g^2$ $(p=1)$.

If $b'=c'=0$, the line $x=0$, $z=0$ counts for a tacnodal generator. The symbol is $\overline{2c_1^2}+c_2^2+[c_3^2]+2g^2$ $(p=1)$, the cubic being of genus 1. If $b'^2=4a'c'$, there is an ordinary double generator in the plane of the conic, and a singular tacnodal one in the plane $x=0$. The symbol is $\overline{2c_1^2}+c_2^2+[c_3^2]+\overline{2g^2}+g^2$ $(p=0)$. Finally, if $b'=c'=a=0$, the S_6 is unicursal and the singular line, which touches c_2^2 counts for three double generators. The residual nodal curve is a cubic having a double point at the point of contact of c_2 and g. The symbol is $\overline{2c_1^2}+c_2^2+[c_3]^2+\overline{3g^2}$ $(p=0)$.

15. In the same manner, let a (1, 2) correspondence between the planes through c_1 and the points on c_1 be given, and let c_1 cut a twisted cubic c_3 in one point. Then will π cut c_3 in two points.

Let $x=\lambda$, $y=\lambda^2$, $z=\lambda^3$ be c_3, and let $z=0$, $x=0$ be the line.

$$x=mz, \qquad m=\frac{a'\mu^2+b'\mu+c'}{a\mu^2+b\mu+c}.$$

There are two positions of the plane such that the two points associated with one of the values of μ in this plane are collinear, hence the line joining them is a double generator. The symbol is $\overline{2c_1^2}+2c_3^2+2g^2$, both cubics being space curves.

If the twisted cubic be replaced by a plane nodal cubic, the two double generators will lie in one plane, meeting in the node.

Let $wy^2=x(x-w)^2$, $z=0$ be $[c_3]$,

$$\begin{aligned} y_1 &= mx_1, \\ w_1y_1^2 &= x_1(x_1-w_1)^2, \end{aligned} \qquad \frac{a'\mu^2+b'\mu+b}{a\mu^2+b\mu}.$$

The equation is

$$2\left[(ay-a'x)\big((w-x)\,2zx-z^2y\big)+(by-b'x)\,c_3\right]\left[-b'xz^2-2zx\,(x-w)\right] \\ -\big((ay-a'x)\,z+c'c_3\big)^2=0.$$

The residual nodal curve is another $[c_3]$ having its node at (0, 0, 0, 1). The symbol is $\overline{2c_1^2} + 2[c_3]^2 + 2g^2$.

The dual of all the unicursal S_6's with a tacnodal directrix are of one kind. The nodal curve consists of a fourfold directrix and two generators, the former counting as 8. Any plane section has two distinct tacnodes on it.

§3.—*Triple Directrix Line.*

(a). *No Double Generator.*

16. Let c_1 cut c_5 in a double point, and be in (1, 1) correspondence with it. c_1 is now simple directrix and double generator. The residual is a c_7^2, having two triple points, and cutting c_1 four times. [This was type XXXV.] No sub-forms can exist, since no double tangents can be drawn. Scrolls of this type are necessarily of the [3, 3] type, since c_5 must have a double point.

If c_1 and c_5 be in (2, 1) correspondence, the same type as before results except that now c_1 is double directrix and simple generator (old type XLV).

If c_1 and c_5 be in (3, 1) correspondence, such that both values of λ at the node correspond to the μ of that point, c_1 is a triple directrix (old type XXIII). This last type can be generated by developables of the form

$$\begin{aligned} at^3 + bt^2 + ct + d &= 0, \\ xt^3 + y &= 0. \end{aligned}$$

Using the same notation as on p. 73, Vol. XXV,

$$\begin{aligned} \psi_1 &= x\{(dx - ay)c + b^2y\} = 0, \\ \psi_2 &= \quad (dx - ay)^2 + bcxy = 0, \\ \psi_3 &= y\{(dx - ay)b - c^2x\} = 0. \end{aligned}$$

If, in particular, b, c pass through (0, 0, 0, 1), one of the triple points on c_7 is on c_1 at this point.

If $a = a_1x + a_2y + a_3z + a_4w$ and similarly for the other terms, $c_4 = 0$, $b_4 = 0$, and it is no restriction if $c_3 : c_1 = b_2 : b_1$. If

$$a_4b_1 + b_2d_4 = 0,$$

the other triple point on c_7 becomes consecutive to the first one. All the branches have a common tangent at this point.

(b). *One Double Generator.*

17. If the two values of λ at a second node on c_5 correspond to the same value of μ in the (1, 2) correspondence, a double generator exists (type XLVI). The residual sextic cuts c_1 in three points. In particular, the c_6 may be replaced by a triple conic. If

$$\lambda = \frac{y}{x}, \quad \mu = \frac{yz}{yw - x^2}$$

and λ, μ be connected by a (2, 3) correspondence having 0, 0 for a double element, the resulting S_6 will have the symbol $(c_1^2 + g') + g^2 + c_2^3$.

The same two forms will exist if the correspondence be between a triple line and a plane c_5 or a triple c_2. In case a multiple point of c_6 lies on c_1, the g^2 may pass through this point, or the other one. The discussion is fully given by Bergstedt. If $b \equiv c$, the $c_6^2 \equiv c_3^2$.

(c). *Two Double Generators.*

18. If, in the (1, 2) correspondence between c_1, c_5, two branch-points exist at nodes, the symbol becomes $(c_1^2 + g') + 2g^2 + c_5^2$ (XLVII). Similarly for the other case. The two g^2 cannot intersect.

(d). *Three Double Generators.*

19. The line c_1 must now be a triple directrix, since a (1, 2) correspondence can only have two branch-points. The residual is a c_4 of the second kind. Finally, two of these may become tacnodal or the three may become oscnodal or all may unite in a triple generator. The last form requires that all three values of λ at the triple point of c_5 correspond to the same value of μ on c_1. It may also be shown as follows: Since any plane section of S_6 containing g^3 is a rational cubic, let

$$x = 1 - \lambda^2, \quad y = \lambda(1 - \lambda^2), \quad z = 0, \quad w = 1,$$
$$x = a, \quad y = b, \quad z = \mu, \quad w = 1,$$
$$\mu f_3(\lambda) + 1 - \lambda^2 - a + m\lambda(1 - \lambda^2) - mb = 0.$$

The S_6 is of the form required, since the three values of λ corresponding to $\mu=0$ are collinear.

(e). *Four Double Generators.*

20. As S_6 of this form must belong to a linear congruence, only two types will be considered, as particular cases which were omitted before.

Given a c_5 having two distinct nodes P_1, P_2 and four consecutive double points at P. Draw a line c_1 through P_2 not in plane of c_5. Make the pencil whose vertex is P_1 projective with range on l in such a way that P_1P_2 corresponds to P_2. Connect each point of l with the three points in which the corresponding line through P_1 cuts c_5. The line l will be a triple line on the sextic scroll; there will be four consecutive double generators at P, hence the residual nodal line will be another triple directrix. An arbitrary plane through the singular generator will cut a quartic curve having an oscnode on the singular line. If, instead of four consecutive nodes at P, a tacnode and a simple branch passing through it be chosen, the singular generator will count as triple and consecutive nodal. Any plane passed through the line will have a nodal cubic as residual section. Similarly, scrolls of higher order can have all their double generators consecutive or coincident. The elimination is rational in each case.

§4.—*Fourfold Directrix Line.*

21. The enumeration given in my previous list is complete. They may all be generated by c_1 and c_6, the former passing through a triple point on the latter. When the point is a cusp, the residual quartic curve has a double point on c_1, and similarly for the cubic when a double generator exists. Sixteen forms exist. The correspondence may be (1, 1) without any corresponding element, or (2, 1) with a simple self-corresponding element, or (3, 1) with a double (really two simple elements, defining a branch-point) element, or, finally, (4, 1) with all three self-corresponding elements. In all except the first form one or two double generators may exist.

22. The dual of a tacnodal directrix is a fourfold line, such that the two generators which lie in any plane through the line intersect on the line.

Let x_1, y_1, 0, w_1 be a point on the c_5,

$$x^2yw^2 + x\mu_3(x, y)w + \mu_5 = 0, \quad z = 0,$$

which has a tacnodal point and a simple branch at the point (0, 0, 0, 1). Connect the point to the point $(0, 0, \mu, 1)$ on c_1,

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{\mu w - z}{\mu w_1}$$

and let

$$y_1 = mx_1, \quad \mu = \frac{m}{am^2 + bm + c}.$$

The equation of the scroll is

$$(ax^2z + bxy(z - w) + cy^2z)^2 + (ax^2z + bxy(z - w) + cy^2z)u_3(x, y) + yu_5(x, y) = 0.$$

The line x, y is a double tacnodal directrix through each point of which pass four generators. Any plane section cuts this line in a pair of distinct tacnodal branches, hence it is equivalent to a nodal curve of order 8, when c_5 has no further singular points $p = 2$, and the surface has no other nodal line. If $u_3 = u_1v_2$ and $u_5 = u_1^2v_3$, one double generator exists, and if $u_3 = u_1v_1w_1$, $u_5 = u_1^2v_1^2t_1$, two such may exist. The former is of genus 1 and the latter of genus 0.

By replacing c_5 by a c_3 of the form

$$yw^2 + u_2w + u_3 = 0, \quad z = 0$$

and letting

$$y_1 = mx_1, \quad \mu = m^2,$$

a S_6 is obtained in which the directrix has the same form as before. The equation is

$$(wy^2 - x^2z)^2 + y(wy^2 - x^2z)u_2(x, y) + y^3u_3(x, y) = 0,$$

which is of genus 1. When $u_2 = u_1v_1$ and $u_3 = u_1^2w_1$, a double generator appears. The line $y = 0$, $z = 0$ is a cuspidal generator. If $a = 0$ and $b = 0$, the line $x = 0$, $y = 0$ is a double generator.

Four types of the unicursal S_6 exist. When the two double points on c_5 are distinct, the generators passing through them may be skew or intersect on the directrix, when the two planes form a pair in the involution. Similarly, if the two nodes form a tacnode, the generators do not lie in the same plane unless the plane be a double plane of the involution.

The second and fourth of these surfaces can, therefore, be generated by a (1, 4) correspondence between a conic and a straight line which it does not intersect.

If $\mu = \frac{\phi_2(m)}{\psi^2(m)}$, the most general form of the equation is

$$(\phi_2(x, y)\, w - z\psi_2(x, y))^2 = xy\, [\phi_2(x, y)]^2.$$

If ϕ_2 is a square, the double generators are tacnodal.

5. *Fivefold Directrix Line.*

23. The twelve forms mentioned complete the list. They can all be generated by c_1 and c_5 in $(k, 1)$ correspondence, $k = 1$ to 5.

B. *Scrolls of Genus One.*

24. The S_6 of genus 1, which were omitted, are similar to the corresponding unicursal scrolls having a double rectilinear directrix. If, on p. 80, $e = c = 0$, the common chord is a double generator. If $a = 1$, $f = bg$, a c_1^2, also appears. Its symbol is $c_1^2 + [c_3^2] + 2c_2^2 + g^2$.

This scroll can also be generated as follows:

Given the non-singular cubic c_3,

$$y^2 w = x^3 - x w^3, \quad z = 0$$

and the line c_1, $x = 0$, $y = 0$ passing through the intersection of c_3 and one of its harmonic polars. Join the point $(0, 0, z_2, w_2)$ on c_1 to $(x_1, y_1, 0, w_1)$ on c_3.

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{wz_2 - zw_3}{z_2},$$

from which
$$y^2 x_1 = x^2 (x_1^2 - w_1^2), \quad xz_2 = x_1 z_2 w - x_1 z.$$

Let the parameters x_1, z_2 be connected by the relation

$$x_1 = z_2^2.$$

The resulting S_6 will have the equation

$$\begin{vmatrix} x^2 & 0 & -y^2 & 0 & -x & 0 \\ 0 & x^2 & 0 & -y^2 & z & x \\ w & -z & -x & 0 & 0 & 0 \\ 0 & w & -z & -x & 0 & 0 \\ 0 & 0 & w & -z & -1 & 0 \\ 0 & 0 & 0 & w & 0 & 1 \end{vmatrix} = 0.$$

c_1 and c_3 will each be double curves. There is one $g^2, x = 0,\ w = 0$. In the planes $z \pm iw = 0$ lie the c_2^2, $2x^2 \pm i(y^2 + z^2) = 0$, hence the symbol is $c_1^2 + [c_3^2] + g^2 + 2c_2^2$. Since the section made by the plane $w = 0$ is g^2 and the tacnodal quartic

$$z^4 - x^4 + z^2y^2 = 0,$$

hence S_6 is of genus 1, and has no further nodal curve.

Similarly, if a c_1 and an elliptic c_5 be put in (2, 1) correspondence with a tacnodal generator, the symbol becomes $c_1^2 + \overline{2g^2} + c_6^2$. The c_1 may lie on a double tangent, or on two or three. In the latter case c_6 is replaced by a c_4 and a c_2, the latter being tangent to the singular generator. The c_4 is of the first kind and cuts c_2 in two points not on $\overline{2g^2}$.

My form VII on p. 96 is incorrect. This S_6 is unicursal and contains a g^3. The forms with a tacnodal directrix are most easily derived from the dual.

C. *Scrolls of Genus Greater than One.*

25. The only form omitted in $p = 2$ is $c_1^2 + c_4^2 + [c_3^2]$. The cubic is non-singular and cuts c_1. c_4 is of genus 1 and cuts c_3 in four points. The surface can be generated by an elliptic (2, 2) correspondence between c_1 and c_3, the point of intersection being a double self-corresponding point. This form may be generated as follows:

Given a non-singular cubic curve and a point P upon it, not on a harmonic polar. Let Q be the first tangential of P. Let a range on a straight line passing through P, but not lying in the plane of the cubic, be made projective with the pencil whose vertex is Q in the plane of the cubic. The line PQ of the pencil should correspond to the point P of the range. Finally, with each point of the range should be associated the point symmetric to it with regard to P. Lines joining the points of this double range to each of the points in which the corresponding line of the pencil cut the cubic will be a S_6 having the symbol $c_1^2 + [c_3^2] + c_4^2$ and of genus 2. E. g. given $y^2w = x^3 - 12xw^2$, $z = 0$ and $p \equiv (2, 4, 0, 1)$. Then $Q \equiv (4, 4, 0, 1)$. The correspondence is expressed by

$$\begin{aligned} x &= -2w, & x - 4w &= \lambda(x - 4w), \\ y &= 4w, & \lambda &= \mu^2, \\ z &= \mu w. \end{aligned}$$

By writing

$$y^2w - x^3 + 12x = c_3, \quad 24zw^2 - 24zxw - 6xz - 8yzw - zy^2 = a,$$
$$4z^2\{2(y-4) - 3x\} = b,$$

the equation of S_6 becomes, multiplied by $[y - 4w + az]^2$,

$$[(4w - y)c + b(4w - x)]^2 = [a(4w - y) - 6bz] \, . \, [a(x - 4w) - 6cz].$$

The quartic curve is of the first kind, and does not cut c_1.

Similarly, if P be a point of inflexion, and Q coincide with P, the four generators which lie in the same plane through c_1 will intersect in pairs on c_1. Let

$y_1w_1^2 = x_1^3 - x_1y_1w_1, \quad z = 0,$ be c_3,

$y_1 = \mu^2 x_1$ be the pencil Q,

$\dfrac{x}{x_1} = \dfrac{y}{y_1} = \dfrac{\mu - z}{\mu}$ be the line joining corresponding points.

The equation of the surface is

$$x\sqrt{x - y} = \sqrt{yw} - \sqrt{xz},$$

or, rationalized,

$$[x^2(x - y) - yw^2 - xz^2]^2 = 4w^2z^2xy.$$

The section of the surface made by $w = 0$ is a cubic of exactly the same form as the given one. The directrix $x = 0$, $y = 0$ is not a generator, but every plane

section will have a tacnode at the point of intersection with it, and three other nodes on each cubic. The surface is of genus 2 and symbol $c_{1,2}^2 + 2\,[c_3^2]$.

26. The general dual of the form treated in §13 may be generated as follows: To the range $(0, 0, \mu, 1)$ corresponds the axial pencil $\mu z + w = 0$, and to the points of c_5 correspond the planes

$$x_1 x + y_1 y + w = 0,$$

wherein
$$x_1^2 y_1 w_1^2 + x_1 u_3(x_1, y_1)\, w_1 + u_5(x_1, y_1) = 0,$$

$$y_1 = m x_1, \quad \mu = \frac{m}{am^2 + bm + c}.$$

From these equations we obtain

$$awm^2 + (bw + z)\, m + cw = 0,$$
$$m\,(x + my)^2 - wu_3(m)(x + my) + w^2 u_5(m) = 0.$$

The m eliminant is the surface required, after dividing out the extraneous factor w^3. The form of this equation is sufficiently general to include all the scrolls having a tacnodal directrix, but it is easier to obtain the dual of each particular case directly. Of those of genus 1 and simple residual nodal curve, two forms exist: one double generator and c_6^2 or a tacnodal double generator and a c_5^2. The other forms have already been derived directly.

Table of Forms.

27. In the following list of types, group A includes all the unicursal ones, while groups B, C contain only those which were not included in the previous lists. The notation for the symbol of the type is the same as that employed before, the number in the next column refers to the paragraph of the present paper in which the corresponding type is derived. Finally, the numbers 2, 3 in the last column of A give the class of the simplest developable which all the generators of the scroll touch. Those having the number 2 are of [2, 4] type, while those marked 3 are of [3, 3] type. The enumeration is now believed to be complete.

A.

1, 2, 3	$c_1^1 + c_{10,6}^2$	2	2
4, 5, 6	$c_1^1 + c_{10,3}^2$	3	3
7	$c_1^1 + 2c_{5,3}^2$	2	2
8	$c_1^1 + 2c_{5,3}^2$	3	3
9, 10	$c_1^2 + c_{9,6}^3$	4	2
11	$c_1^2 + c_{6,4}^2 + [c_{3,2}^2]$	4	2
12	$c_1^2 + 3\,[c_{3,2}^2]$	4	2
13, 14	$c_1^2 + c_{9,3}^2$	5	3
15	$c_1^2 + c_{6,2}^2 + c_3^2$	5	3
16	$c_1^2 + 3c_3^2$	5	3
17–24	replace c_1^2 by $(c_1^1 + g^1)$ in 9–16	5	
25, 26	$c_1^2 + c_{8,5}^2 + g^2$	6	2
27	$c_1^2 + c_{6,4}^2 + c_2^2 + g^2$	6	2
28	$c_1^2 + c_{5,3}^2 + [c_3^2] + g^2$	6	2
29	$c_1^2 + c_2^2 + 2\,[c_3^2] + g^2$	6	2
30 31	$c_1^2 + c_{8,3}^2 + g^2$	7	3
32	$c_1^2 + c_{6,3}^2 + c_2^2 + g^2$	7	3
33	$c_1^2 + c_{5,2}^2 + c_3^2 + g^2$	7	3
34	$c_1^2 + c_2^2 + 2c_3^2 + g^2$	7	3
35, 36	$c_1^2 + c_{7,4}^2 + 2g^2$	8	2
37, 38	$c_1^2 + c_{7,4}^2 + \overline{2g^2}$	8	2
39, 40	$c_1^2 + c_{7,3}^2 + 2g^2$	9	3
41, 42	$c_1^2 + c_{7,3}^2 + \overline{2g^2}$	9	3
43	$c_1^2 + c_{4,2}^2 + [c_{3,2}^2] + 2g^2$	9	2
44	$c_1^2 + c_{4,2}^2 + [c_{3,2}^2] + \overline{2g^2}$	9	2
45	$c_1^2 + c_{5,3}^2 + c_2^2 + \overline{2g^2}$	9	2
46	$c_1^2 + [c_{3,2}^2] + 2c_2^2 + \overline{2g^2}$	9	2
47	$c_1^2 + c_4^2 + c_3^2 + 2g^2$	10	3
48	$c_1^2 + c_{5,2}^2 + c_2^2 + 2g^2$	10	3
49	$c_1^2 + c_3^2 + 2c_2^2 + 2g^2$	10	3
50	$c_1^2 + c_{5,2}^2 + c_2^2 + \overline{2g^2}$	10	3
51	$c_1^2 + c_3^2 + 2c_2^2 + \overline{2g^2}$	10	3

52	$c_1^2 + c_{6,3}^2 + \overline{3g^2}$	11	2
53	$c_1^2 + c_{6,3}^2 + g^2 + \overline{2g^2}$	11	3
54	$c_1^2 + c_4^2 + c_2^2 + g^2 + \overline{2g^2}$	11	3
55	$c_1^2 + c_{4,2}^2 + c_2^2 + \overline{3g^2}$	11	2
56	$c_1^2 + 3c_2^2 + g^2 + \overline{2g^2}$	11	3
57	$c_1^2 + c_4^2 + \overline{3g^2}$	12	2
58	$\overline{2c_1^2} + c_5^2 + \overline{3g^2}$	13	2
59	$\overline{2c_1^2} + c_6^2 + 2g^2$	13	2
60	$\overline{2c_1^2} + c_6^2 + 2g^2$	13	3
61	$\overline{2c_1^2} + c_5^2 + g^2 + \overline{2g^2}$	13	3
62	$\overline{2c_1^2} + c_4^2 + 2 \,.\, \overline{2g^2}$	13	3
63	$\overline{2c_1^2} + 2c_2^2 + 2 \,.\, 2g^2$	13	3
64	$\overline{2c_1^2} + c_2^2 + [c_{3,2}^2] + \overline{2g^2} + g^2$	14	3
65	$\overline{2c_1^2} + c_2^2 + [c_{3,2}^2] + \overline{3g^2}$	14	2
66	$\overline{2c_1^2} + 2c_3^2 + 2g^2$	15	3
67	$\overline{2c_1^2} + 2\,[c_{3,2}^2] + 2g^2$	15	2
68	$(c_1^1 + g^2) + c_{7,3}^2$	16	3
69	$(c_1^2 + g^1) + c_{7,3}^2$	16	3
70	$c_1^3 + c_{7,3}^2$	16	3
71	$(c_1^2 + g^1) + c_{6,2}^2 + g^2$	17	3
72	$(c_1^3 + g^1) + c_2^3 + g^2$	17	3
73	$c_1^3 + c_{6,2}^2 + g^2$	17	3
74	$c_1^3 + c_2^3 + g^2$	17	3
75	$(c_1^2 + g^1) + c_{5,2}^2 + 2g^2$	18	3
76	$c_1^3 + c_{5,2}^2 + 2g^2$	18	3
77	$c_1^3 + c_4^2 + 3g^2$	19	3
78	$c_1^3 + c_4^2 + \overline{2g^2} + g^2$	19	3
79	$c_1^3 + c_4^2 + \overline{3g^2}$	19	3
80	$c_1^3 + c_4^2 + g^3$	19	3
81	$c_1^2 + c_1^4 + \overline{4g^2}$	20	2
82–97	$c_1^4 +$ . . . . given before.	21	2
98, 99	$\overline{2c_1^4} + 2g^2$	22	2
100, 101	$\overline{2c_1^4} + \overline{2g^2}$	22	2

102–113	Fivefold line. Vol. XXV, p. 83	23	1
114–122	Obvious forms contained in a linear congruence, not previously mentioned in this paper.		

B.

1	$c_1^2 + [c_3^2] + 2c_2^2 + g^2$	24	
2, 3	$c_1^2 + c_6^2 + \overline{2g^2}$	24	
4	$c_1^2 + c_2^2 + c_4^2 + \overline{2g^2}$	24	
5, 6	$\overline{2c_1^2} + c_{6,2}^2 + g^2$	26	
7, 8	$\overline{2c_1^2} + c_5^2 + 2g^2$	26	
9	$\overline{2c_1^2} + 2c_2^2 + c_4^2 + g^2$	14	
10	$\overline{2c_1^2} + [c_3^2] + c_2^2 + \overline{2g^2}$	14	
11	$2c_1^4 + g^2$	22	37 forms.

C.

1	$c_1^2 + c_4^2 + [c_3^2]$	25	
2	$\overline{2c_1^2} + c_6^2$	26	
3	$\overline{2c_1^2} + 2[c_3^2]$	25	
4	$\overline{2c_1^4}$	26	14 forms.
			6 forms, $p > 2$.

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